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# ENDOGENEITY OF ALTERNATING OFFERS IN A BARGAINING GAME

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## Abstract

We investigate an infinite horizon two-person simultaneous offer bargaining game of incomplete information with discounted playoffs. In each period, each player chooses to give in or hold out. The game continues until at least one of the players chooses to give in, at which point agreement has been reached and the game terminates, with an agreement benefit accruing to each player, and a cost to the player (or players) that give in. Players have privately known agreement benefits. Low benefit players have a weakly dominant strategy to hold out forever; high benefit players would be better off giving in if they knew their opponent was planning to hold out forever.

For any discount factor there is a unique Nash equilibrium in which the two players alternate in their willingness to give in, if the players' priors about each others type are sufficiently asymmetric. Second, for almost all priors, this is the unique equilibrium if the discount factor is close enough to one.

**Endogeneity of Alternating Offers**
**in a Bargaining Game**[1]

## 1 INTRODUCTION

There has been a large literature over the last decade studying the bargaining problem when there are costs of delay. Rubinstein [1982] launched this literature with his seminal paper formulating this novel approach for the study of bargaining. His original model was a complete information game with alternating offers for division of a fixed pie. A significant feature of this model is that it has a unique stationary equilibrium with intuitively sensible properties. Subsequent work has extended the model by introducing incomplete information (see Cramton [1984], Chatterjee and Samuelson [1987], Abreu and Gul [1992], Grossman and Perry [1986], Ausbel and Deneckere [1986]). All of these models have assumed the same basic extensive form as is assumed by Rubinstein – namely an alternating offer framework.

In the context of a simple model, we offer a partial answer to the following question. Why should one assume an alternating offer structure in a bargaining game, and how do the results change if we assume a different game form? The problem of choice of extensive form has remained unanswered in the decade since it was first articulated by Fudenberg, Levine and Tirole (1985, p. 73):

> "... the reliance of the noncooperative approach on particular extensive forms poses two problems. First, because the results depend on the extensive form, one needs to argue that the chosen specification is reasonable -- that it is a good approximation to the extensive forms actually played. Second, even if one particular extensive form were used in almost all bargaining, the analysis is incomplete because it has not, at least to-date, begun to address the question of why that extensive form is used. This chapter will consider the first point of extending the class of bargaining games for which we have solutions. The second and harder problem, we will leave unresolved."

In this paper, we investigate a two-person simultaneous offer model of bargaining game with incomplete information. In each period, there is a simultaneous move in

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which each player chooses either to give in or to hold out. The game continues until at least one of the players chooses to give in, at which point agreement is reached and the game ends, with a benefit accruing to each player, and a cost to the player (or players) that gave in. Players have privately known agreement benefits. Low benefit players have a dominant strategy to always hold out, and a high benefit player's best response depends on what the opponent does.

For any discount factor, we find that for asymmetric enough priors over the types of the players, there is a unique Nash equilibrium in which the two players alternate in their willingness to give in. Moreover, for any amount of asymmetry of priors, this is the unique Nash equilibrium for sufficiently high discount factors. These uniqueness results do not require additional restrictions such as stationarity of strategies or sequential rationality. Thus, in a very strong sense, alternating offers arise endogenously, even when the underlying game form has a simultaneous move structure.

The model we analyze can be viewed as a version of the war of attrition game (Maynard Smith 1974) with incomplete information in which one of the types has a dominant strategy to fight forever. Thus it is an incomplete information version of a simple game of timing (Fudenberg and Tirole 1991). The possibility of multiple equilibria with an alternating play structure for complete information wars of attrition is reported in Hendricks and Wilson (1989) for reasons related to the existence of multiple asymmetric equilibrium in a one-shot game of chicken. This paper reports a much stronger result that the *unique* Nash equilibrium necessarily exhibits an alternating play structure if player beliefs are sufficiently asymmetric or if the discount factor is sufficiently close to one.

## 2 A BARGAINING GAME

We consider a two person simultaneous offer bargaining game, in which, at each move, the players can decide to hold out (H) or give in (G). The game ends as soon as the first player decides to give in, with payoffs given in the following matrix.

| | G | | H | |
|---|---|---|---|---|
| G | $b^1$ | $b^2$ | $b^1$ | 1 |
| H | 1 | $b^2$ | 0 | 0 |

Each player discounts payoffs if the game ends on move $\tau$ by a common discount factor $\delta^{\tau-1}$. Here we consider an incomplete information version of the game in which each player has two possible types (values of $b^i$), $b_L$ and $b_H$, where $b_L < 0 < b_H < 1$. The probability that that $b^i = b_H$ equals $r_i$ for player $i$, which is common knowledge. We will also write $b = b_H$.

We represent the infinite horizon game in its reduced normal form. Let $S_1 = S_2 = \{\infty, 0, 1, 2, \ldots, t, \ldots\}$ be the set of pure actions available to each type of each player, where $\infty$ corresponds to never giving in and a nonnegative integer $t$ corresponds to holding out for exactly $t$ rounds before giving in. The set of mixed actions available to each type of each player is $\Sigma^i = \{(\pi_\infty, \pi_0, \pi_1, \ldots): \pi_t \geq 0$ for all $t$, and $\sum_t \pi_t = 1\}$. A mixed strategy for $i$ is a pair $(\pi^i_H, \pi^i_L) \in \Sigma^i \times \Sigma^i$, which specifies a probability distribution over how many periods to hold out, depending on type (H or L). Let $r_i$ be the prior belief that player $i$ is a high benefit type, and $r = (r_1, r_2)$ be the prior belief profile. We assume throughout that $1 > r_i > 0$, $i = 1, 2$. We write $\overline{r}_i = 1 - r_i$ for $i = 1, 2$ to be the corresponding probability of a low benefit type and denote $p_i = r_i\pi^i_H + \overline{r}_i\pi^i_L \in \Sigma^i$. Then for any $p \in \Sigma^1 \times \Sigma^2$, the expected payoff to player $i$ is:

$$M_i^H(p) = \sum_{t=0}^{\infty} \pi_H^{it} \left[\sum_{l<k} \delta^l p_{jl} + \sum_{l \geq t} \delta^l b p_{jl}\right] \quad \text{if } b_i = b$$

$$= \sum_{t=0}^{\infty} \pi_L^{it} \left[\sum_{l<k} \delta^l p_{jl} + \sum_{l \geq t} \delta^l b_L p_{jl}\right] \quad \text{if } b_i = b_L$$

Similarly, the payoff to $i$ for using a pure strategy $t$ when $j$ uses $p_j$ is:

$$M_i^H(t, p_j) = \delta^t b + \sum_{l<t} (\delta^l - \delta^t b) p_{jl} \qquad \text{if } b_i = b$$

$$= \delta^t b_L + \sum_{l<t} (\delta^l - \delta^t b_L) p_{jl} \qquad \text{if } b_i = b_L$$

## 3 SOLUTION

First, it is clear from the normal form that since $b_L < 0$, strategy 0 is strictly dominated for all $b_L$ types, and iterated elimination of strictly dominated strategies leaves only "$\infty$" for $b_L$ since $r \neq 0$. So $\pi_L^{i\infty} = 1$ in any Nash equilibrium. Thus, we just need to determine the strategy of the high benefit types. Similarly, one can show $\pi_H^{i\infty} = 0$ in any Nash equilibrium. This is proved as Lemma 0 in the Appendix. Henceforth we drop type subscripts and suppress this $\infty$ action, replacing $p_{i\infty}$ with (1 - $r_i$) and $p_{it} = \pi_H^{it}$, $t = 0, 1, 2, \ldots$.

Define

$$R = \frac{b - \delta b}{1 - \delta b}$$

and

$$Q = \frac{1 + \delta}{\delta} R = \frac{(1 - \delta^2)b}{\delta(1 - \delta b)}.$$

We also use the notation

$$\overline{R} = 1 - R = \frac{1 - b}{1 - \delta b}$$

and

$$\overline{Q} = 1 - Q = \frac{\delta - b}{\delta(1 - \delta b)}$$

We first prove some properties of the equilibrium strategy.

***Lemma 1:*** Let $p = (p_1, p_2)$ be a Nash equilibrium. If $p_{jt} = 0$ for some $t$, then $p_{i,t+1} = p_{j,t+2} = 0$.

***Proof:*** Assume that $p_{jt} = 0$. Then

$$\begin{aligned} M_i(t+1, p_j) &= \textstyle\sum_{t < l} \delta^{t+1} b p_{jl} + \sum_{t \geq l} [\delta^l r_j + \delta^{t+1} \overline{r}_j b] p_{jl} \\ &= \textstyle\sum_{t \leq l} \delta^{t+1} b p_{jl} + \sum_{t > l} [\delta^l r_j + \delta^{t+1} \overline{r}_j b] p_{jl} \\ &< \textstyle\sum_{t \leq l} \delta^t b p_{jl} + \sum_{t > l} [\delta^l r_j + \delta^t \overline{r}_j b] p_{jl} = M_i(t, p_j) \end{aligned}$$

It follows that for player $i$, pure strategy $t$ is better against $p_j$ than pure strategy $t+1$. Hence, in equilibrium, strategy $t+1$ cannot be adopted with positive probability. $\square$

From Lemma 0 and Lemma 1, it follows that any Nash equilibrium will have at most two phases: a phase in which both players make offers with positive probability every round (*the simultaneous offer phase*) followed by a phase in which players alternate in their willingness to make offers (*the alternating offer phase*). Either phase could be empty, but if both phases exist in an equilibrium, the alternating offer phase must always follow the simultaneous offer phase.

***Lemma 2:*** If $p = (p_1, p_2)$ is a Nash equilibrium, and $t$ is any integer, then

*a.* $p_{it} > 0$ and $p_{i,t+1} > 0 \Rightarrow p_{jt} = R\,[\frac{1}{r_j} - \sum_{l=0}^{t-1} p_{jl}] = \frac{R}{r_j}\overline{R}^t$

*b.* $p_{jt} < R\,[\frac{1}{r_j} - \sum_{l=0}^{t-1} p_{jl}] \Rightarrow p_{i,t+1} = 0$

*c.* $p_{jt} > R\,[\frac{1}{r_j} - \sum_{l=0}^{t-1} p_{jl}] \Rightarrow p_{it} = 0$

***Proof:*** Assume that $p_{ik} > 0$, for $k = t, t+1$. Then it must be the case that $M_i(t+1, p_j) = M_i(t, p_j)$. So

$$M_i(t+1, p_j) \geq M_i(t, p_j)$$

$$\Leftrightarrow \delta^{t+1}b + r_j \sum_{l=0}^{t} p_{jl}(\delta^l - \delta^{t+1}b) \geq \delta^t b + r_j \sum_{l=0}^{t-1} p_{jl}(\delta^l - \delta^t b)$$

$$\Leftrightarrow p_{jt} r_j \delta^t (1 - \delta b) + r_j \delta^t (1-\delta) b \sum_{l=0}^{t-1} p_{jl} \geq b\delta^t(1-\delta)$$

$$\Leftrightarrow p_{jt} \geq \frac{(1-\delta)b}{1-\delta b}\,[\frac{1}{r_j} - \sum_{l=0}^{t-1} p_{jl}] = R\,[\frac{1}{r_j} - \sum_{l=0}^{t-1} p_{jl}]$$

So

$$M_i(t+1, p_j) \geq M_i(t, p_j) \Leftrightarrow p_{jt} \geq R\,[\frac{1}{r_j} - \sum_{l=0}^{t-1} p_{jl}]$$

Similarly

$$M_i(t+1, p_j) \leq M_i(t, p_j) \Leftrightarrow p_{jt} \leq R\,[\frac{1}{r_j} - \sum_{l=0}^{t-1} p_{jl}].$$

Using the fact that all strategies in the support of the mixed strategy must have equal payoffs, the result (b), (c), and the first part of (a) follow immediately. To obtain the remainder of (a), note that if $p_{it} > 0$ and $p_{i,t+1} > 0$, then by Lemma 1, it follows that $p_{ik} > 0$ for all $k \leq t+1$. In particular, $p_{i0} > 0$. It follows that $p_{j0} = R/r_j$. For $t \geq 1$,

$$p_{jt} = R\,[\tfrac{1}{r_j} - \textstyle\sum_{l=0}^{t-2} p_{jl} - p_{j,t-1}] = p_{j,t-1} - Rp_{j,t-1} = \overline{R}p_{j,t-1}$$

So, by induction on $t$,

$$p_{jt} = \tfrac{R}{r_j}\overline{R}^t \qquad \square$$

***Lemma 3:*** Let $p = (p_1, p_2)$ be a Nash equilibrium. For any $t \geq 1$,

*a.* $p_{jt} \leq Q[\tfrac{1}{r_j} - \sum_{l=0}^{t-1} p_{jl}] - \tfrac{1}{\delta}\overline{R}p_{j,t-1}$.

*b.* $p_{jt} < Q[\tfrac{1}{r_j} - \sum_{l=0}^{t-1} p_{jl}] - \tfrac{1}{\delta}\overline{R}p_{j,t-1} \Rightarrow p_{i,t+1} = 0$.

*c.* $p_{i,t+1} > 0 \Rightarrow p_{jt} = Q[\tfrac{1}{r_j} - \sum_{l=0}^{t-1} p_{jl}] - \tfrac{1}{\delta}\overline{R}p_{j,t-1}$.

*d.* If $t \geq 3$, $p_{i,t+1} > 0$ and $p_{j,t-1} = 0 \Rightarrow p_{jt} = \overline{Q}p_{j,t-2} + \tfrac{1}{\delta}\overline{R}p_{j,t-3}$

***Proof:*** Assume that $p_{i,t+1} > 0$. Then it follows from Lemma 1 that $p_{i,t-1} > 0$. Hence, $M_i(t+1, p_j) \geq M_i(t-1, p_j)$. So

$$M_i(t+1, p_j) \geq M_i(t-1, p_j)$$

$$\Leftrightarrow \delta^{t+1}b + \textstyle\sum_{l=0}^{t} p_{jl}r_j(\delta^l - \delta^{t+1}b) \geq \delta^{t-1}b + \sum_{l=0}^{t-2} p_{jl}r_j(\delta^l - \delta^{t-1}b)$$

$$\Leftrightarrow p_{jt}r_j\delta^t(1-\delta b) + p_{j,t-1}r_j\delta^{t-1}(1-\delta^2 b)$$

$$+ r_j\delta^{t-1}(1-\delta^2)b\textstyle\sum_{l=0}^{t-2} p_{jl} \geq b\delta^{t-1}(1-\delta^2)$$

$$\Leftrightarrow p_{jt} \geq \frac{(1-\delta^2)b}{\delta(1-\delta b)}[\frac{1}{r_j} - \textstyle\sum_{l=0}^{t-2} p_{jl}] - \frac{1-\delta^2 b}{\delta(1-\delta b)}p_{j,t-1}$$

$$= Q[\tfrac{1}{r_j} - \textstyle\sum_{l=0}^{t-2} p_{jl}] - (Q + \tfrac{1}{\delta}\overline{R})p_{j,t-1}$$

$$= Q[\tfrac{1}{r_j} - \textstyle\sum_{l=0}^{t-1} p_{jl}] - \tfrac{1}{\delta}\overline{R}p_{j,t-1}.$$

So

$$M_i(t+1,p_j) \geq M_i(t-1,p_j) \Leftrightarrow p_{jt} \geq Q[\tfrac{1}{r_j} - \Sigma_{l=0}^{t-1} p_{jl}] - \tfrac{1}{\delta}\overline{R}p_{j,t-1}.$$

Similarly,

$$M_i(t+1,p_j) \leq M_i(t-1,p_j) \Leftrightarrow p_{jt} \leq Q[\tfrac{1}{r_j} - \Sigma_{l=0}^{t-1} p_{jl}] - \tfrac{1}{\delta}\overline{R}p_{j,t-1}$$

So (a) and (c) follow from these inequalities, by an argument similar to that in Lemma 2. Note that Lemma 1 implies that $M_i(t+1,p_j) \leq M_i(t-1,p_j)$

Now if $p_{j,t-1} = 0$, then we get

$$\begin{aligned} p_{jt} &= Q[\tfrac{1}{r_j} - \Sigma_{l=0}^{t-2} p_{jl}] \\ &= Q[\tfrac{1}{r_j} - \Sigma_{l=0}^{t-3} p_{jl}] - \tfrac{1}{\delta}\overline{R}p_{j,t-3} - Qp_{j,t-2} + \tfrac{1}{\delta}\overline{R}p_{j,t-3} \\ &= \overline{Q}p_{j,t-2} + \tfrac{1}{\delta}\overline{R}p_{j,t-3} \end{aligned}$$

□

Putting together the above lemmas, we can completely characterize the Nash equilibrium of the game. Proposition 1 characterizes equilibria that begin with a non-empty simultaneous offer phase. Proposition 2 characterizes the alternating offer equilibria.

***Proposition 1:*** Let $p = (p_1, p_2)$ be a Nash equilibrium satisfying $p_{10} > 0$ and $p_{20} > 0$. Then it must be of one of the following two forms:

***Form 1:*** **(*Full simultaneous offer equilibrium*)** There exists an integer $M \geq 0$ such for $i = 1,2$

$$p_{it} = \frac{R}{r_i}\overline{R}^t \qquad \text{for } 0 \leq t \leq M-1$$

$$p_{it} = \frac{\overline{R}^t - \overline{r}_i}{r_i} \qquad \text{for } t = M$$

$$p_{it} = 0 \qquad \text{otherwise}$$

where $M$ satisfies

$$\overline{R}^{M+1} \leq \overline{r}_i \leq \overline{R}^M$$

***Form 2:* (*Partial simultaneous offer equilibrium*)** The players can be labeled $\{i,j\} = \{1,2\}$ and there exists an integer $M \geq 0$, and an integer $J \geq 1$, such that

$$p_{it} = \frac{R}{r_i}\overline{R}^t \qquad \text{for } 0 \leq t \leq M$$

$$p_{it} = \frac{Q}{r_i}\overline{R}^{M+1}\overline{Q}^{l-1} \qquad \text{for } t = M + 2l,\ 1 \leq l \leq J-1$$

$$p_{i,M+2J} = \frac{1}{r_i}[\overline{R}^{M+1}\overline{Q}^{J-1} - \overline{r}_i]$$

$$p_{it} = 0 \qquad \text{otherwise}$$

and

$$p_{jt} = \frac{R}{r_j}\overline{R}^t \qquad \text{for } 0 \leq t \leq M-1$$

$$p_{jM} = \frac{\overline{r}_j - \overline{R}^M\overline{Q}^J}{r_j\overline{Q}^{J-1}(\frac{1}{\delta} - \overline{R})}$$

$$p_{jM+1} = \frac{\overline{R}^M\overline{Q}^{J-1} - \overline{r}_j}{r_j\overline{Q}^{J-1}} - p_{jM}$$

$$p_{jt} = \frac{\overline{r}_jQ}{r_j\overline{Q}^{J-l}} \qquad \text{for } t = M + 2l + 1,\ 1 \leq l \leq J-1$$

$$p_{it} = 0 \qquad \text{otherwise}$$

where $M \geq 0$ and $J > 0$ satisfy

$$\overline{R}^{M+1}\overline{Q}^J \leq \overline{r}_i \leq \overline{R}^{M+1}\overline{Q}^{J-1}$$

$$\overline{R}^M\overline{Q}^J \leq \overline{r}_j \leq \overline{R}^{M+2}\overline{Q}^{J-1}$$

***Proof:*** See Appendix

***Proposition 2:*** Let $p = (p_1, p_2)$ be a Nash equilibrium satisfying $p_{i0} = 0$. Then it must be of the following form:

***Form 3:*** (***Alternating offer equilibrium***) There exists an integer $J \geq 1$, such that

$$p_{it} = \frac{Q}{\overline{r}_i}\overline{Q}^{l-1} \qquad \text{for } t = 2l - 1,\ 1 \leq l \leq J - 1$$

$$p_{i,2J} = \frac{1}{\overline{r}_i}[\overline{Q}^{J-1} - \overline{r}_i]$$

$$p_{it} = 0 \qquad \text{otherwise}$$

and

$$p_{j0} = \frac{\overline{Q}^{J-1} - \overline{r}_j}{r_j\overline{Q}^{J-1}}$$

$$p_{jt} = \frac{\overline{r}_j Q}{r_j\overline{Q}^{J-l}} \qquad \text{for } t = 2l,\ 1 \leq l \leq J - 1$$

$$p_{jt} = 0 \qquad \text{otherwise}$$

where $J > 0$ satisfies

$$\overline{Q}^J \leq \overline{r}_i \leq \overline{Q}^{J-1}$$

$$0 \leq \overline{r}_j \leq \overline{Q}^{J-1}\overline{R}$$

***Proof:*** See Appendix.

***Corollary 1:*** If $\overline{r}_j > \overline{R}$, then there is a pure strategy equilibrium. There are two cases:

Case I: $\overline{r}_i > \overline{R}$. Here we have a unique equilibrium with $p_{i0} = p_{j0} = 1$.

Case II: $\overline{r}_i < \overline{R}$. Here we have a unique equilibrium with $p_{i0} = 1, p_{j1} = 1$.

***Proof:*** This follows directly from Propositions 1 and 2. Since $\overline{r}_j > \overline{R}$, it follows that $\overline{r}_j > \overline{R}^k$ and $\overline{r}_j > \overline{Q}^k$ for all $k \geq 1$. Therefore, the only values that can be chosen for $M$ and $J$ are $M = 0, J = 0$ (in case 1) in which case

$$\overline{R}^{M+1} < \overline{r}_i < \overline{R}^M$$
$$\overline{R}^{M+1} < \overline{r}_j < \overline{R}^M$$

or $M = -1,\ J = 1$ (case 3) in which case

$$\overline{Q}^J < \overline{R} < \overline{r}_i < \overline{R} < \overline{Q}^{J-1}$$
$$\overline{R}\,\overline{Q}^{J-1} = \overline{R} < \overline{r}_j < 1 = \overline{Q}^{J-1}$$

The only way we can have a pure strategy equilibrium is if $M = 0$ and $J = 0$, or if if and only if

$$\overline{R} < \overline{r}_i < 1$$
$$\overline{R} < \overline{r}_j < 1$$

and $M = -1,\ J = 1$ if and only if

$$\overline{Q} < \overline{r}_i < 1$$
$$\overline{R} < \overline{r}_j < 1$$

□

***Corollary 2:*** If $\overline{r}_1 < \overline{Q}$ and $\overline{r}_2 < \overline{Q}$, then there is no pure strategy equilibrium.

***Proof:*** This follows directly from Proposition 1. The only way we can have a pure strategy equilibrium is if $M = 0$ and $J = 0$, or $M = -1,\ J = 1$. But in the first case, we must have $\overline{R} < \overline{r}_i < 1$ and $\overline{R} < \overline{r}_j < 1$. But $\overline{Q} < \overline{R} \Rightarrow \overline{Q} < \overline{r}_i$ and $\overline{Q} < \overline{r}_j$, a contradiction. Similarly, in the second case, we have $\overline{Q} < \overline{r}_i < 1$ and $\overline{R} < \overline{r}_j < 1$. But again this implies $\overline{Q} < \overline{r}_j$, a contradiction. □

We now define a set $\mathcal{N} \subseteq [0,1] \times [0,1]$ which we call the *necktie* as follows:

$$\mathcal{N} = \ \bar{r} = (\bar{r}_1, \bar{r}_2) \subseteq [0,1]^2\colon \bar{r}_1 \geq \frac{\overline{Q}}{\overline{R}}\bar{r}_2, \geq \bar{r}_2 \geq \frac{\overline{Q}}{\overline{R}}\bar{r}_1\} \qquad (*)$$

***Theorem:*** There is a Nash equilibrium satisfying $p_{10} > 0$ and $p_{20} > 0$ only if $r \in \mathcal{N}$. If $r \notin \mathcal{N}$, then there is a unique Nash equilibrium. The equilibrium is an *alternating offer equilibrium* such that if $r_j > r_i$, then $p_{i,2k} = p_{j,2k+1} = 0$ for all $k \geq 0$.

***Proof:*** Assume that $p_{10} > 0$ and $p_{20} > 0$. Then by proposition 1, there are two cases. If the equilibrium is a full simultaneous offer equilibrium, then for some integer $M \geq 0$,

$$\overline{R}^{M+1} \leq \bar{r}_i \leq \overline{R}^M$$
$$\overline{R}^{M+1} \leq \bar{r}_j \leq \overline{R}^M.$$

It is easily verified that $\overline{R}^2 > \overline{Q}$, from which it follows that

$$\bar{r}_i \geq \overline{R}^{M+1} \geq \overline{R}\bar{r}_j > \frac{\overline{Q}}{\overline{R}}\bar{r}_j$$

A similar argument establishes that

$$\bar{r}_j > \frac{\overline{Q}}{\overline{R}}\bar{r}_i.$$

On the other hand, if the equilibrium is a partial simultaneous offer equilibrium, then there are integers $M \geq 0$, and $J > 0$ such that

$$\overline{R}^{M+1}\overline{Q}^J \leq \bar{r}_i \leq \overline{R}^{M+1}\overline{Q}^{J-1}$$

$$\overline{R}^M\overline{Q}^J \leq \bar{r}_j \leq \overline{R}^{M+2}\overline{Q}^{J-1}$$

It follows that

$$\bar{r}_i \geq \overline{R}^{M+1}\overline{Q}^J = \frac{\overline{Q}}{\overline{R}}\overline{R}^{M+2}\overline{Q}^{J-1} \geq \frac{\overline{Q}}{\overline{R}}\bar{r}_j$$

Similarly,

$$\bar{r}_j \geq \overline{R}^M\overline{Q}^J = \frac{\overline{Q}}{\overline{R}}\overline{R}^{M+1}\overline{Q}^{J-1} \geq \frac{\overline{Q}}{\overline{R}}\bar{r}_i.$$

To show the second part of the proposition, suppose that $r \notin \mathcal{N}$. Assume that $r_i > r_j$, so that $\bar{r}_i < \bar{r}_j$. Then since $r \notin \mathcal{N}$, $\bar{r}_i < (\overline{Q}/\overline{R})\bar{r}_j$. By the above argument, there is no equilibrium that begins with a simultaneous offer. Further, by proposition 2, for any alternating offer equilibrium with $p_{i0} = 0$ there must be an integer $J > 0$ such that

$\overline{Q}^J \leq \overline{r}_i$, and $\overline{r}_j \leq \overline{Q}^{J-1}\overline{R}$. But this implies that $\overline{r}_i \geq \overline{Q}^J = (\overline{Q}/\overline{R})\overline{Q}^{J-1}\overline{R} \geq (\overline{Q}/\overline{R})\overline{r}_j$, which is a contradiction. Since there must exist at least one equilibrium, there must be an alternating offer equilibrium with $p_{j0} = 0$, and by proposition 2, it follows that it is unique. A similar argument establishes that if $r \notin \mathcal{N}$, with $r_i < r_j$, then there is a unique alternating offer equilibrium with $p_{i0} = 0$. □

The theorem shows that outside the necktie the unique equilibrium is an alternating offer equilibrium with *no simultaneous offers phase.* The simple structure of the necktie, implies several properties, which are stated formally as Corollaries 3, 4 and 5. Informally, they are stated as follows:

1. If the players are sufficiently patient, then alternating offers is the unique equilibrium.

2. If both players are sufficiently likely to be high benefit types, then alternating offers is the unique equilibrium.

3. If the players' beliefs about each other are sufficiently different, then alternating offers is the unique equilibrium.

***Corollary 3.*** Fix $\{r_1, r_2, b\}$ with $r_1 \neq r_2$ and $0 < b < 1$. Then there exists $\widehat{\delta}$ such that the unique equilibrium is alternating offers for all $\delta \geq \widehat{\delta}$.

***Corollary 4.*** Fix $\{\Delta, \delta, b\}$ with $\Delta \in (0,1)$ and $0 < b < \delta < 1$. Then there exists $\widehat{r}_2 < 1 - \Delta$ such that the unique equilibrium is alternating offers, for all $(r_1, r_2)$ such that $r_2 \in (\widehat{r}_2, 1\text{-}\Delta)$ and $1 \geq r_1 \geq r_2 + \Delta$.

***Corollary 5.*** Fix $\{\delta, b\}$ with $0 < b < \delta < 1$. Then there exists $\widehat{\Delta} \in (0,1)$ such that the unique equilibrium is alternating offers for all $(r_1, r_2) \in (0,1)^2$ such that $r_1 - r_2 \geq \widehat{\Delta}$.

All of these results follow directly from the definition of $\mathcal{N}$, which says that all points in the necktie lie in the quadrilateral bounded by the points (1, 1), (0, 0), $(0, \frac{b(1-\delta)}{\delta(1-b)})$, $(\frac{b(1-\delta)}{\delta(1-b)}, 0)$.

These can be illustrated using a "belief diagram", similar to what appears in Chatterjee and Samuelson (1987). Figures 1-3 represent the common knowledge beliefs of the players, with $r_1$ on the horizontal axis and $r_2$ on the vertical axis. The diagonal represents the case of symmetric beliefs where $r_1 = r_2$. Below of the diagonal, player 1 is believed to be the more likely of the two players to have a high benefit, and the reverse is true above the diagonal. The previous section characterized the equilibrium correspondence mapping the unit square of beliefs into Bayesian equilibria. The equilibrium for the case of $\delta =$ 75 and $b = .5$ are illustrated in Figures 1 and 2, each of which divides the unit square into several regions. The necktie is the quadrilateral bounded by the slanted lines near the diagonal. In the shaded region of Figure 1 there is an equilibrium which is *alternating offers* and this equilibrium is unique outside the necktie. Figure 2 shows the structure of equilibria inside the necktie. In the gray shaded areas, a simultaneous offer phase precedes the alternating offer phase. The black areas have no alternating offer phase. Otherwise, the length of the simultaneous offer phase is indicated by the darkness of shading, with darker shades representing longer simultaneous offer phases. The areas of overlap, implying multiple equilibria, are indicated in black. Figure 3 shows the effect of increasing $\delta$ to .90. The main thing to notice is that the necktie becomes narrower. In the limit it converges to the diagonal, which implies the property stated in the Corollary 1.

## 4 SOLUTION IN BEHAVIOR STRATEGIES

Define

$$f_{it} = \sum_{l \leq t} p_{il}$$
$$g_{it} = \sum_{l \geq t} p_{il}$$

to be the cumulative spent and unspent probability for player $i$ at move $t$. Then the behavioral strategy for the high benefit type player $i$ is

$$q_{it} = \frac{p_{it}}{g_{it}}.$$

Thus, $q_{it}$ is the conditional probability a high type will give in given that neither player has given in yet. For the alternating offer solution, we get for $1 \le l \le J-1$

$$g_{i,2l-1} = g_{i,2l} = 1 - \Sigma_{k<2l} p_{ik} = 1 - \frac{Q}{r_i}\Sigma_{k=1}^{l-1}\overline{Q}^{k-1}$$

$$= 1 - \frac{1-\overline{Q}^{l-1}}{r_i} = \frac{1}{r_i}[\overline{Q}^{l-1} - \overline{r}_i]$$

$$q_{i,2l-1} = \frac{p_{i,2l-1}}{g_{i,2l-1}} = \frac{Q\overline{Q}^{l-1}}{\overline{Q}^{l-1} - \overline{r}_i} = \frac{Q}{1-(\overline{r}_i/\overline{Q}^{l-1})} = \frac{Q}{\widehat{r}_{il}},$$

where

$$\widehat{r}_{il} = 1 - (\overline{r}_i/\overline{Q}^{l-1}),$$

is the posterior of $r_i$ at the $l^{th}$ inning: in other words,

$$\widehat{r}_{i1} = 1 - \overline{r}_i = r_i,$$

and given $\widehat{r}_{it}$, we get

$$\widehat{r}_{i,t+1} = 1 - \frac{\overline{r}_i}{\overline{Q}^t} = \frac{\widehat{r}_{it} - Q}{\overline{Q}} = \frac{\widehat{r}_{it}(1-q_{it})}{1-\widehat{r}_{it}q_{it}} = \frac{\widehat{r}_{it}(1-q_{it})}{\widehat{r}_{it}(1-q_{it}) + (1-\widehat{r}_{it})}$$

which is simply Bayes rule for updating $\widehat{r}_{it}$ given the behavioral strategy $q_{it}$.

Similarly, for player $j$, we get $g_{j0} = 1$, and for $t = 2l$, $1 \le l \le J-1$

$$g_{j,2l} = \Sigma_{k \ge 2l} p_{ik} = \frac{\overline{r}_j Q}{r_j \overline{Q}^{J-l}} \Sigma_{k=0}^{J-l-1} \overline{Q}^k = \frac{\overline{r}_j}{r_j \overline{Q}^{J-l}}[1 - \overline{Q}^{J-l}]$$

So

$$q_{j0} = p_{j0} = \frac{\overline{Q}^{J-1} - \overline{r}_j}{r_j \overline{Q}^{J-1}}.$$

$$q_{j,2l} = \frac{p_{j,2l}}{g_{j,2l}} = \frac{\overline{r}_j Q}{r_j \overline{Q}^{J-l}} \frac{r_j \overline{Q}^{J-l}}{\overline{r}_j[1-\overline{Q}^{J-l}]} = \frac{Q}{1-\overline{Q}^{J-l}}$$

$$\widehat{\overline{r}}_{j1} = \frac{\overline{r}_j}{1 - r_j q_{i0}} = \frac{\overline{r}_j}{1 - \dfrac{\overline{Q}^{J-1} - \overline{r}_j}{\overline{Q}^{J-1}}} = \frac{\overline{r}_j \overline{Q}^{J-1}}{\overline{Q}^{J-1} - \overline{Q}^{J-1} + \overline{r}_j} = \overline{Q}^{J-1}$$

$$\widehat{\overline{r}}_{j,l+1} = \frac{\widehat{\overline{r}}_{jl}}{1 - \widehat{r}_{jl} q_{j,2l}} = \frac{\overline{Q}^{J-l}}{1 - (1 - \overline{Q}^{J-l}) \dfrac{Q}{1 - \overline{Q}^{J-l}}}$$

$$= \frac{\overline{Q}^{J-l}}{1-Q} = \overline{Q}^{J-(l+1)}$$

which is simply Bayes rule for updating $\widehat{r}_{jt}$ given the behavioral strategy $q_{jt}$.

## 5 CONCLUSION

The main point of this paper was to show that assumption of "alternating offers," commonly used in theoretical models of bargaining, arises endogenously as the *unique* equilibrium of a simultaneous-move infinite horizon bargaining game, if players are sufficiently patient or if their beliefs are sufficiently disparate. The model we use to demonstrate this is a very simple one, with two types of each player, and only two available strategies for each player. A natural next question to investigate is whether this main feature of our model is robust to more general specifications. In addition to allowing for a larger variety of types and continuous strategy spaces (corresponding to "real" bargaining), it would also be interesting to extend the result to games with a richer payoff structure, particularly a payoff structure in which there is less surplus lost when both parties simultaneously concede. Results elsewhere in the literature on games of timing (Fudenberg and Tirole (1991) and Hendricks and Wilson (1989)) and on bargaining games specifically (Abreu and Gul (1992)), suggest that such generalizations are possible.

## REFERENCES


Abreu, D., and F. Gul, "Bargaining with obstinate players", ***mimeo***, (December, 1992).

Ausbel, L. and R. Deneckere, "A direct mechanism characterization of sequential bargaining with one-sided incomplete information", ***Journal of Economic Theory***, 48 (1989): 18-46.

Chatterjee, K., and L. Samuelson, "Bargaining with two sided incomplete information: An infinite horizon model with alternating offers", ***Review of Economic Studies***, 54 (1987): 175-192.

Cramton, P. C., "Bargaining with incomplete information: An infinite horizon model with continuous uncertainty", ***Review of Economic Studies***, 51 (1984): 579-594.

Fudenberg, D. and D. Levine and J. Tirole, "Infinite horizon models in bargaining with one-sided incomplete information", in ***Game Theoretic Models of Bargaining***, (A. Roth, editor), New York: Cambridge University Press, 1985: 73-98.

Fudenberg, D. and J. Tirole, ***Game Theory,*** Cambridge: Massachusetts Institute of Technology Press, 1991.

Grossman, S. and M. Perry, "Sequential bargaining under asymmetric information," ***Journal of Economic Theory***, 39 (1986): 119-54.

Hendricks, K. and C. Wilson, "The war of attrition in discrete time," mimeo, 1989.

Rubinstein, A., "Perfect equilibrium in a bargaining model", ***Econometrica***, 50 (1982): 97-109.

Rubinstein, A., "A bargaining model under incomplete information", ***Econometrica***, 53 (1985): 1151-1172.

Smith, Maynard Jr., "The theory of games and evolution in animal conflicts," ***Journal of Theoretical Biology***, 47 (1974): 209-21

## APPENDIX

**Lemma 0.** Let $\pi$ be a Nash equilibrium. Then $\pi_L^{i\infty} = 1$ and $\pi_H^{i\infty} = 0$.

**Proof:** Consider type $L$. We show that iterated elimination of strictly dominant strategies exhibits all actions except $\infty$ for type $L$. Giving in immediately (action 0) yields an expected payoff of $b_L < 0$, which is strictly dominated by a waiting forever (action $\infty$) which guarantees a payoff of at least 0. Therefore eliminate all strategies with $\pi_L^{i0} > 0$. Given $\pi_L^{i0} = 0$, it follows that giving in on a second round (action 1) is strictly dominated by waiting forever ($\infty$). Therefore eliminate all strategies with $\pi_L^{iL} > 0$. In a similar way, one can show that if $\pi_L^{ik} = 0$, $\forall k = 0, 1, \ldots$,t then action $t + 1$ is strictly dominated by $\infty$. By induction, it follows that $\pi_L^{it} = 0$, $\forall t < \infty$. Therefore in any Nash equilibrium, $\pi_L^{i\infty} = 1$, $i = 1, 2$.

Next consider type $H$ of player 1. Let $\pi$ be a Nash equilibrium. From above, $\pi_L^{2\infty} = 1$. Therefore,

$$M_1^H(\infty, p_\alpha) \;\leq\; r_j \sum_{t=0}^{\infty} \delta^t \pi_L^{2t}, \text{ and}$$

$$M_1^H(k, p_2) = (1\text{-}\, r_j)\; \delta^k b + r_j \left[\sum_{t=0}^{k-1} \delta^t \pi_H^{2t} + b \sum_{t=k}^{\infty} \delta^k \pi_H^{2t}\right]$$

So, $M_1^H(k, p_2) > M_1^H(\infty, \pi^2)$ if

$$(1 - r_j)\delta^k b + r_j b \sum_{t=k}^{\infty} \delta^k \pi_H^{2t} > r_j \sum_{b=k}^{\infty} \delta^t \pi_H^{2t}$$

$$\Leftrightarrow 1 - r_j b + r_j b \sum_{t=k}^{\infty} \pi_H^{2t} > \sum_{t=k}^{\infty} \delta^{t-k} \pi_H^{2t}.$$

As $k \to \infty$, the LHS converges to $1 - r_j b > 0$ and the RHS converges to 0. Therefore, for sufficiently high $k$, $M_1^H(k, p^2) > M_1^H(\infty, p^2)$.

Therefore, any strategy with $\pi_H^{1\infty} > 0$ is never a weak best response to $\pi^2$. Since $\pi$ is a Nash equilibrium, it follows that $\pi_H^{1\infty} = 0$. By a similar argument, $\pi_H^{2\infty} = 0$.

***Proof of Proposition 1:*** Let $p$ be a Nash equilibrium with $p_{10} > 0$ and $p_{20} > 0$. Let $M \geq 0$ be the last round in which both players mix, and $L \geq 0$ be the number of rounds in which only one player mixes. There are two cases:

*Case 1:* $L = 0$

By Lemma 2, we have

$$p_{it} = \frac{R}{\overline{r}_i}\overline{R}^t \qquad \text{for } 0 \leq t \leq M-1$$

If $L = 0$, then we must have

$$p_{iM} = 1 - \sum_{l=0}^{M-1} p_{il} = 1 - \frac{R}{\overline{r}_i}\sum_{l=0}^{M-1}\overline{R}^l = 1 - \frac{1-\overline{R}^M}{\overline{r}_i} = \frac{\overline{R}^M - \overline{r}_i}{\overline{r}_i}.$$

For this to be an equilibrium, we must have $0 \leq p_{iM}$, which implies that $\overline{r}_i \leq \overline{R}^M$. Also, by Lemma 2, we must have

$$p_{iM} \leq \frac{R}{\overline{r}_i}\overline{R}^M \Leftrightarrow \overline{R}^M - \overline{r}_i \leq R\overline{R}^M \Leftrightarrow \overline{R}^{M+1} \leq \overline{r}_i.$$

Hence, we have shown that

$$\overline{R}^{M+1} \leq \overline{r}_i \leq \overline{R}^M$$

A similar argument for player $j$ establishes that the mixed strategy for $j$ takes the same form, and that $\overline{r}_j$ must satisfy the inequality

$$\overline{R}^{M+1} \leq \overline{r}_j \leq \overline{R}^M$$

Thus, the equilibrium must be of the first form.

*Case 2:* $L \geq 1$

Without loss of generality, assume that $p_{i,M+1} = 0$ and $p_{j,M+1} > 0$. Then, by Lemma 1, it follows that $p_{i,M+2k-1} = p_{j,M+2k} = 0$ for all $1 \leq k$. By Lemma 2 (a), we have

$$p_{it} = \frac{R}{\overline{r}_i}\overline{R}^t \qquad \text{for } 0 \leq t \leq M$$

As long as $k \geq 1$ and $2k+1 \leq L$, it follows that $p_{j,M+2k+1} \neq 0$. So by Lemma 3 (c), since $p_{i,M+2k-1} = 0$,

$$p_{i,M+2k} = Q[\frac{1}{\overline{r}_i} - \sum_{l=0}^{M+2k-2} p_{il}]$$

If $k = 1$, then

$$p_{i,M+2k} = p_{i,M+2} = Q[\frac{1}{\bar{r}_i} - \Sigma_{l=0}^{M} p_{il}]$$

$$= Q[\frac{1}{\bar{r}_i} - \Sigma_{l=0}^{M-1} p_{il}] - Qp_{iM}$$

$$= \frac{Q}{R} R[\frac{1}{\bar{r}_i} - \Sigma_{l=0}^{M-1} p_{il}] - Qp_{iM}$$

$$= \frac{Q}{R} p_{iM} - Qp_{iM} = \frac{Q}{R}\overline{R} p_{iM}$$

$$= \frac{Q}{R}\overline{R}\frac{R}{\overline{r}_i}\overline{R}^M = \frac{Q}{\overline{r}_i}\overline{R}^{M+1}$$

For $k > 1$, we have $p_{i,M+2k-1} = p_{i,M+2k-3} = 0$. So using Lemma 3 (d) (with $i$ playing the role of $j$) , we get

$$p_{i,M+2k} = \overline{Q} p_{i,M+2k-2} = \frac{Q}{\overline{r}_i}\overline{R}^{M+1}\overline{Q}^{k-1}$$

Let $J$ be the smallest integer for which $2J+1 > L$. Then we must have $p_{i,M+2J+k} = 0$ for all $k \geq 1$. Hence, since $\sum_k p_{ik} = 1$, it must be that

$$p_{i,M+2J} = 1 - \sum_{k < M+2J} p_{ik}$$

But

$$\sum_{k < M+2J} p_{ik} = \frac{R}{\overline{r}_i}\sum_{k=0}^{M}\overline{R}^k + \frac{Q}{\overline{r}_i}\overline{R}^{M+1}\sum_{l=0}^{J-2}\overline{Q}^t$$

$$= \frac{1}{\overline{r}_i}\{[1 - \overline{R}^{M+1}] + \overline{R}^{M+1}[1 - \overline{Q}^{J-1}]\}$$

$$= \frac{1}{\overline{r}_i}[1 - \overline{R}^{M+1}\overline{Q}^{J-1}]$$

So

$$p_{i,M+2J} = 1 - \sum_{k < M+2J} p_{ik} = 1 - \frac{1}{\overline{r}_i}[1 - \overline{R}^{M+1}\overline{Q}^{J-1}]$$

$$= \frac{\overline{R}^{M+1}\overline{Q}^{J-1} - \overline{r}_i}{\overline{r}_i}$$

Since $p_{i,M+2J} \geq 0$, it follows that $\overline{r}_i \leq \overline{R}^{M+1}\overline{Q}^{J-1}$. Also, from Lemma 3 (a), and since $p_{j,M+2J+1} = 0$,

$$p_{i,M+2J} \leq \frac{Q}{\overline{r}_i}\overline{R}^{M+1}\overline{Q}^{J-1} \Leftrightarrow \overline{R}^{M+1}\overline{Q}^{J-1} - \overline{r}_i \leq Q\overline{R}^{M+1}\overline{Q}^{J-1}$$

$$\Leftrightarrow \overline{R}^{M+1}\overline{Q}^{J-1} - Q\overline{R}^{M+1}\overline{Q}^{J-1} \leq \overline{r}_i$$

$$\Leftrightarrow \overline{R}^{M+1}\overline{Q}^{J} \leq \overline{r}_i$$

It follows that

$$\Leftrightarrow \overline{R}^{M+1}\overline{Q}^{J} \leq \overline{r}_i \leq \overline{R}^{M+1}\overline{Q}^{J-1}.$$

We note that generically, the above weak inequalities will be strict, implying that $p_{i,M+2J} > 0$, and that $L = 2J$. For the remainder of the proof we assume we are in this generic case.

Next, for player $j$, we must have, by Lemma 2, that

$$p_{jt} = \frac{R}{\overline{r}_j}\overline{R}^t \qquad \text{for } 0 \leq t \leq M-1$$

Now consider the last round of mixing for player $j$: $t = M + 2J - 1$. Now if $J > 1$, then by Lemma 3, since $p_{i,M+2J} > 0$ and $p_{j,t-1} = p_{j,M+2J} = 0$, it follows from Lemma 3 that

$$p_{jt} = Q[\tfrac{1}{r_j} - \textstyle\sum_{l=0}^{t-1} p_{jl}] - \tfrac{1}{\delta}\overline{R}p_{j,t-1} = Q[\tfrac{1}{r_j} - 1 + p_{jt}] - \tfrac{1}{\delta}\overline{R}p_{j,t-1}$$

$$\Leftrightarrow p_{jt}\overline{Q} = \frac{Q\overline{r}_j}{r_j} - \tfrac{1}{\delta}\overline{R}p_{j,t-1} = \frac{Q\overline{r}_j}{r_j}$$

$$\Leftrightarrow p_{jt} = \frac{\overline{r}_j Q}{r_j \overline{Q}}$$

By induction on $l$, as long as $1 < l < J$, we get, by Lemma 3, that

$$p_{j,M+2l-1} = \frac{1}{\overline{Q}}\, p_{j,M+2l+1} = \frac{\overline{r}_j Q}{r_j \overline{Q}^{J-l+1}}.$$

Finally, for $l = 1$, we get that $p_{j,M+2l-1} = p_{j,M+1}$. If $L > 2$, then by Lemma 3,

$$p_{j,M+3} = Q[\tfrac{1}{r_j} - \textstyle\sum_{l=0}^{M+2} p_{jl}] - \tfrac{1}{\delta}\overline{R}p_{j,M+2}$$

$$= Q[\tfrac{1}{r_j} - \textstyle\sum_{l=0}^{M+1} p_{jl}]$$

$$= Q[\tfrac{1}{r_j} - \textstyle\sum_{l=0}^{M-2} p_{jl}] - Qp_{j,M-1} - Qp_{j,M} - Qp_{j,M+1}$$

$$= \frac{Q}{R}R[\frac{1}{r_j} - \Sigma_{l=0}^{M-2} p_{jl}] - Qp_{j,M-1} - Qp_{j,M} - Qp_{j,M+1}$$

$$= Q[\frac{\overline{R}}{R}]p_{j,M-1} - Qp_{j,M} - Qp_{j,M+1}$$

$$\Leftrightarrow \quad Qp_{j,M+1} = Q[\frac{\overline{R}}{R}]p_{j,M-1} - Qp_{j,M} - p_{j,M+3}$$

$$\Leftrightarrow \quad p_{j,M+1} = [\frac{\overline{R}}{R}]p_{j,M-1} - p_{j,M} - \frac{1}{Q}p_{j,M+3}$$

$$= [\frac{\overline{R}}{R}]\frac{R}{r_j}\overline{R}^{M-1} - \frac{1}{Q}\frac{Q\overline{r}_j}{r_j\overline{Q}^{J-1}} - p_{j,M}$$

$$= \frac{1}{r_j}\overline{R}^M - \frac{\overline{r}_j}{r_j\overline{Q}^{J-1}} - p_{j,M} \qquad (*)$$

Also, as in the proof of Lemma 2, it follows that

$$p_{j,M+1} = Q[\frac{1}{r_j} - \Sigma_{l=0}^{M} p_{jl}] - \frac{1}{\delta}\overline{R}p_{j,M}$$

$$= Q[\frac{1}{r_j} - \Sigma_{l=0}^{M-1} p_{jl}] - (Q + \frac{1}{\delta}\overline{R})p_{j,M}$$

$$= Q[\frac{1}{r_j} - \Sigma_{l=0}^{M-1} p_{jl}] - \frac{1}{\delta}\overline{R}p_{j,M-2} + \frac{1}{\delta}\overline{R}p_{j,M-2} - (Q + \frac{1}{\delta}\overline{R})p_{j,M}$$

$$= Q[\frac{1}{r_j} - \Sigma_{l=0}^{M-2} p_{jl}] - \frac{1}{\delta}\overline{R}p_{j,M-2} - Qp_{j,M-1}$$

$$+ \frac{1}{\delta}\overline{R}p_{j,M-2} - (Q + \frac{1}{\delta}\overline{R})p_{j,M}$$

$$= \overline{Q}p_{j,M-1} + \frac{1}{\delta}\overline{R}p_{j,M-2} - (Q + \frac{1}{\delta}\overline{R})p_{jM}$$

$$= \overline{Q}\frac{R}{r_j}\overline{R}^{M-1} + \frac{1}{\delta}\overline{R}\frac{R}{r_j}\overline{R}^{M-2}$$

$$- (Q + \frac{1}{\delta}\overline{R})p_{jM}$$

$$= \frac{R}{r_j}\overline{R}^{M-1}[\overline{Q} + \frac{1}{\delta}] - (Q + \frac{1}{\delta}\overline{R})p_{jM}$$

$$= \frac{R}{r_j}\overline{R}^{M-1}[\frac{1+\delta}{\delta} - Q] - (Q + \frac{1}{\delta}\overline{R})p_{jM}$$

$$= \frac{1}{r_j}\overline{R}^{M-1}[Q - QR] - (Q + \frac{1}{\delta}\overline{R})p_{jM}$$

$$= \frac{Q}{r_j}\overline{R}^{M} - (Q + \frac{1}{\delta}\overline{R})p_{jM}$$

$$= \frac{Q}{r_j}\overline{R}^{M} - (R + \frac{1}{\delta})p_{jM} \qquad (**)$$

But from $(*)$ and $(**)$ above, we get

$$\frac{Q}{r_j}\overline{R}^{M} - (R + \frac{1}{\delta})p_{jM} = \frac{1}{r_j}\overline{R}^{M} - \frac{\overline{r}_j}{r_j\overline{Q}^{J-1}} - p_{j,M}$$

$$\Leftrightarrow (\frac{1}{\delta} - \overline{R})p_{jM} = \frac{\overline{r}_j}{r_j\overline{Q}^{J-1}} - \frac{1}{r_j}\overline{R}^{M}\overline{Q}$$

$$\Leftrightarrow (\frac{1}{\delta} - \overline{R})p_{jM} = \frac{\overline{r}_j - \overline{R}^{M}\overline{Q}^{J}}{r_j\overline{Q}^{J-1}}$$

$$\Leftrightarrow p_{jM} = \frac{\overline{r}_j - \overline{R}^{M}\overline{Q}^{J}}{r_j\overline{Q}^{J-1}(\frac{1}{\delta} - \overline{R})}$$

And, plugging into $(**)$, we get

$$p_{j,M+1} = \frac{Q}{r_j}\overline{R}^{M} - (\frac{1}{\delta} - \overline{R})p_{jM}$$

$$= \frac{Q}{r_j}\overline{R}^{M} - (R + \frac{1}{\delta})\frac{\overline{r}_j - \overline{R}^{M}\overline{Q}^{J}}{r_j\overline{Q}^{J-1}(\frac{1}{\delta} - \overline{R})}$$

$$= \frac{Q\overline{R}^{M}\overline{Q}^{J-1}(\frac{1}{\delta} - \overline{R})}{r_j\overline{Q}^{J-1}(\frac{1}{\delta} - \overline{R})}$$

$$+\frac{(R+\frac{1}{\delta})\overline{R}^M\overline{Q}^J-(R+\frac{1}{\delta})\overline{r}_j}{r_j\overline{Q}^{J-1}(\frac{1}{\delta}-\overline{R})}$$

$$=\frac{(R+\frac{1}{\delta})\overline{R}^M\overline{Q}^{J-1}-Q\overline{R}^M\overline{Q}^{J-1}-(R+\frac{1}{\delta})\overline{r}_j}{r_j\overline{Q}^{J-1}(\frac{1}{\delta}-\overline{R})}$$

$$=\frac{(R+\frac{1}{\delta}-Q)\overline{R}^M\overline{Q}^{J-1}-(R+\frac{1}{\delta})\overline{r}_j}{r_j\overline{Q}^{J-1}(\frac{1}{\delta}-\overline{R})}$$

$$=\frac{(R+\frac{1}{\delta}-1+\overline{Q})\overline{R}^M\overline{Q}^{J-1}-(R+\frac{1}{\delta}-1+1)\overline{r}_j}{r_j\overline{Q}^{J-1}(\frac{1}{\delta}-\overline{R})}$$

$$=\frac{\overline{R}^M\overline{Q}^{J-1}-\overline{r}_j}{r_j\overline{Q}^{J-1}}-\frac{\overline{r}_j-\overline{R}^M\overline{Q}^J}{r_j\overline{Q}^{J-1}(\frac{1}{\delta}-\overline{R})}$$

$$=\frac{\overline{R}^M\overline{Q}^{J-1}-\overline{r}_j}{r_j\overline{Q}^{J-1}}-p_{jM}$$

So we have shown that

$$p_{jM}=\frac{\overline{r}_j-\overline{R}^M\overline{Q}^J}{r_j\overline{Q}^{J-1}(\frac{1}{\delta}-\overline{R})}$$

and

$$p_{jM+1}=\frac{\overline{R}^M\overline{Q}^{J-1}-\overline{r}_j}{r_j\overline{Q}^{J-1}}-p_{jM}$$

Now $p_{jM}\geq 0 \Leftrightarrow \overline{r}_j\geq \overline{R}^M\overline{Q}^J$. Also, by Lemma 2 (c), we must have

$$p_{jM}\leq R[\tfrac{1}{r_j}-\textstyle\sum_{l=0}^{M-1}p_{jl}]=R[\tfrac{1}{r_j}-\sum_{l=0}^{M-2}p_{jl}]-Rp_{j,M-1}=\tfrac{R}{r_j}\overline{R}^M$$

$$\Leftrightarrow \frac{\overline{r}_j-\overline{R}^M\overline{Q}^J}{r_j\overline{Q}^{J-1}(\frac{1}{\delta}-\overline{R})}\leq\frac{R}{r_j}\overline{R}^M$$

$$\overline{r}_j - \overline{R}^M\overline{Q}^J \le R\overline{R}^M\overline{Q}^{J-1}(\tfrac{1}{\delta} - \overline{R})$$

$$\overline{r}_j \le \overline{R}^M\overline{Q}^J + R\overline{R}^M\overline{Q}^{J-1}(\tfrac{1}{\delta} - \overline{R})$$

$$\overline{r}_j \le \overline{R}^M\overline{Q}^{J-1}[\overline{Q} + R(\tfrac{1}{\delta} - \overline{R})] = \overline{R}^M\overline{Q}^{J-1}[1 - \frac{1+\delta}{\delta}r + \frac{R}{\delta} - R + R^2]$$

$$= \overline{R}^M\overline{Q}^{J-1}[1 - 2R + R^2] = \overline{R}^M\overline{Q}^{J-1}(1-R)^2 = \overline{R}^{M+2}\overline{Q}^{J-1}$$

We note that the constraint $p_{jM+1} \ge 0$ is not binding because

$$p_{jM+1} \ge 0 \Leftrightarrow \frac{\overline{R}^M\overline{Q}^{J-1} - \overline{r}_j}{r_j\overline{Q}^{J-1}} \ge p_{jM}$$

$$\Leftrightarrow \frac{\overline{R}^M\overline{Q}^{J-1} - \overline{r}_j}{r_j\overline{Q}^{J-1}} \ge \frac{\overline{r}_j - \overline{R}^M\overline{Q}^J}{r_j\overline{Q}^{J-1}(\frac{1}{\delta} - \overline{R})}$$

$$\Leftrightarrow (\tfrac{1}{\delta} - \overline{R})(\overline{R}^M\overline{Q}^{J-1} - \overline{r}_j) \ge \overline{r}_j - \overline{R}^M\overline{Q}^J$$

$$\Leftrightarrow \tfrac{1}{\delta}\overline{R}^M\overline{Q}^{J-1} - \tfrac{1}{\delta}\overline{r}_j - \overline{R}^{M+1}\overline{Q}^{J-1} + \overline{R}\overline{r}_j \ge \overline{r}_j - \overline{R}^M\overline{Q}^J$$

$$\Leftrightarrow \tfrac{1}{\delta}\overline{R}^M\overline{Q}^{J-1} - \tfrac{1}{\delta}\overline{r}_j + \overline{R}^M\overline{Q}^J - \overline{R}^{M+1}\overline{Q}^{J-1} \ge \overline{r}_j - \overline{R}\overline{r}_j$$

$$\Leftrightarrow \tfrac{1}{\delta}\overline{R}^M\overline{Q}^{J-1} + \overline{R}^M\overline{Q}^J - \overline{R}^{M+1}\overline{Q}^{J-1} \ge \overline{r}_j - \overline{R}\overline{r}_j + \tfrac{1}{\delta}\overline{r}_j$$

$$\Leftrightarrow \overline{R}^M\overline{Q}^{J-1}[\tfrac{1}{\delta} + \overline{Q} - \overline{R}] \ge \overline{r}_j[1 - \overline{R} + \tfrac{1}{\delta}]$$

$$\Leftrightarrow \overline{R}^M\overline{Q}^{J-1}[\tfrac{1}{\delta} + \overline{Q} - \overline{R}] \ge \overline{r}_j[R + \tfrac{1}{\delta}]$$

$$\Leftrightarrow \overline{R}^M\overline{Q}^{J-1}[\tfrac{1}{\delta} + R - Q] \ge \overline{r}_j[R + \tfrac{1}{\delta}]$$

$$\Leftrightarrow \overline{R}^M\overline{Q}^{J-1}[\tfrac{1}{\delta} + R - \frac{1+\delta}{\delta}R] \ge \overline{r}_j[R + \tfrac{1}{\delta}]$$

$$\Leftrightarrow \overline{R}^M\overline{Q}^{J-1}[\frac{1-R}{\delta}] \geq \overline{r}_j[R+\frac{1}{\delta}]$$

$$\Leftrightarrow \overline{R}^{M+1}\overline{Q}^{J-1} \geq \overline{r}_j[\delta R+1]$$

$$\Leftrightarrow \overline{r}_j \leq \frac{\overline{R}^{M+1}\overline{Q}^{J-1}}{\delta R+1} = \frac{\overline{R}^{M+2}\overline{Q}^{J-1}}{\overline{R}(\delta R+1)}$$

But

$$\overline{R}(\delta R+1) = (1-R)(1+2R) \leq (1-R)(1+R) = 1-R^2 \leq 1$$

So we that

$$\overline{R}^{M+2}\overline{Q}^{J-1} \leq \frac{\overline{R}^{M+2}\overline{Q}^{J-1}}{\overline{R}(\delta R+1)}$$

Hence, the restriction on $\overline{r}_j$ is not binding. □

***Proof of Proposition 2:*** The proof is similar to that of proposition 1. So we only verify the inequalities on $\overline{r}_j$ and $\overline{r}_i$. For player $i$, in order for $p_i$ to be an equilibrium strategy, we need to have

$$0 \leq p_{i,2J} \Leftrightarrow \overline{r}_i \leq \overline{Q}^{J-1}$$

Also, by Lemma 2, we need

$$p_{i,2J} \leq \frac{Q}{\overline{r}_i}\overline{Q}^{J-1} \Leftrightarrow \frac{1}{\overline{r}_i}[\overline{Q}^{J-1} - \overline{r}_i] \leq \frac{Q}{\overline{r}_i}\overline{Q}^{J-1}$$

$$\Leftrightarrow \overline{Q}^{J-1} - \overline{r}_i \leq Q\overline{Q}^{J-1}$$

$$\Leftrightarrow \overline{Q}^{J-1} - Q\overline{Q}^{J-1} \leq \overline{r}_i$$

$$\Leftrightarrow \overline{Q}^J \leq \overline{r}_i$$

So we have shown that

$$\overline{Q}^J \leq \overline{r}_i \leq \overline{Q}^{J-1}$$

For Player $j$, since $p_{i0} = 0$ and $p_{i1} > 0$, if follows from Lemma 2b that

$$p_{j0} \geq R\,[\frac{1}{\overline{r}_j} - \sum_{l=0}^{t-1} p_{jl}] = \frac{R}{\overline{r}_j}$$

$$\Leftrightarrow \frac{\overline{Q}^{J-1} - \overline{r}_j}{r_j \overline{Q}^{J-1}} \geq \frac{R}{\overline{r}_j} \Leftrightarrow \overline{Q}^{J-1} - \overline{r}_j \geq R\overline{Q}^{J-1} \Leftrightarrow \overline{Q}^{J-1} - R\overline{Q}^{J-1} \geq \overline{r}_j$$

$$\Leftrightarrow \overline{r}_j \leq \overline{Q}^{J-1}\overline{R}$$

Note that the inequality $p_{j0} \leq 1$ does not impose any additional constraint, as it is always satisfied:

$$p_{j0} \leq 1 \Leftrightarrow \frac{\overline{Q}^{J-1} - \overline{r}_j}{r_j \overline{Q}^{J-1}} \leq 1 \Leftrightarrow \overline{Q}^{J-1} - \overline{r}_j \leq r_j \overline{Q}^{J-1}$$

$$\Leftrightarrow \overline{Q}^{J-1} - r_j \overline{Q}^{J-1} \leq \overline{r}_j \Leftrightarrow \overline{Q}^{J-1} \overline{r}_j \leq \overline{r}_j$$

$$\Leftrightarrow \overline{Q}^{J-1} \leq 1 \qquad \square$$

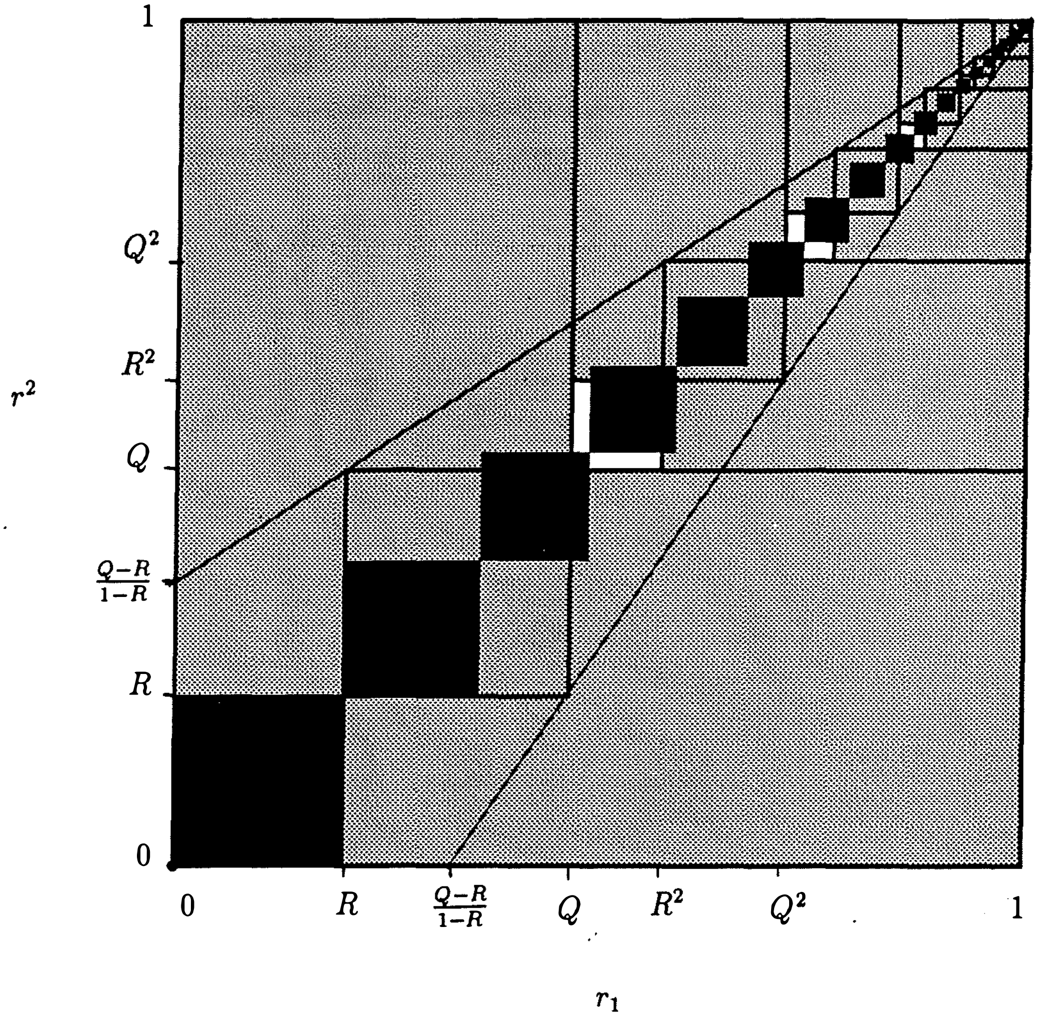


Figure 1. The necktie ($b = .5$, $\delta = .75$) is quadrilateral bounded by $(0,0)$, $(0, \frac{Q-R}{1-R})$, $(1,1)$, and $(\frac{Q-R}{1-R}, 0)$. Alternating offer equilibria exist in shaded areas, and are unique outside the necktie. Simultaneous move equilibria exist in the black-shaded squares.

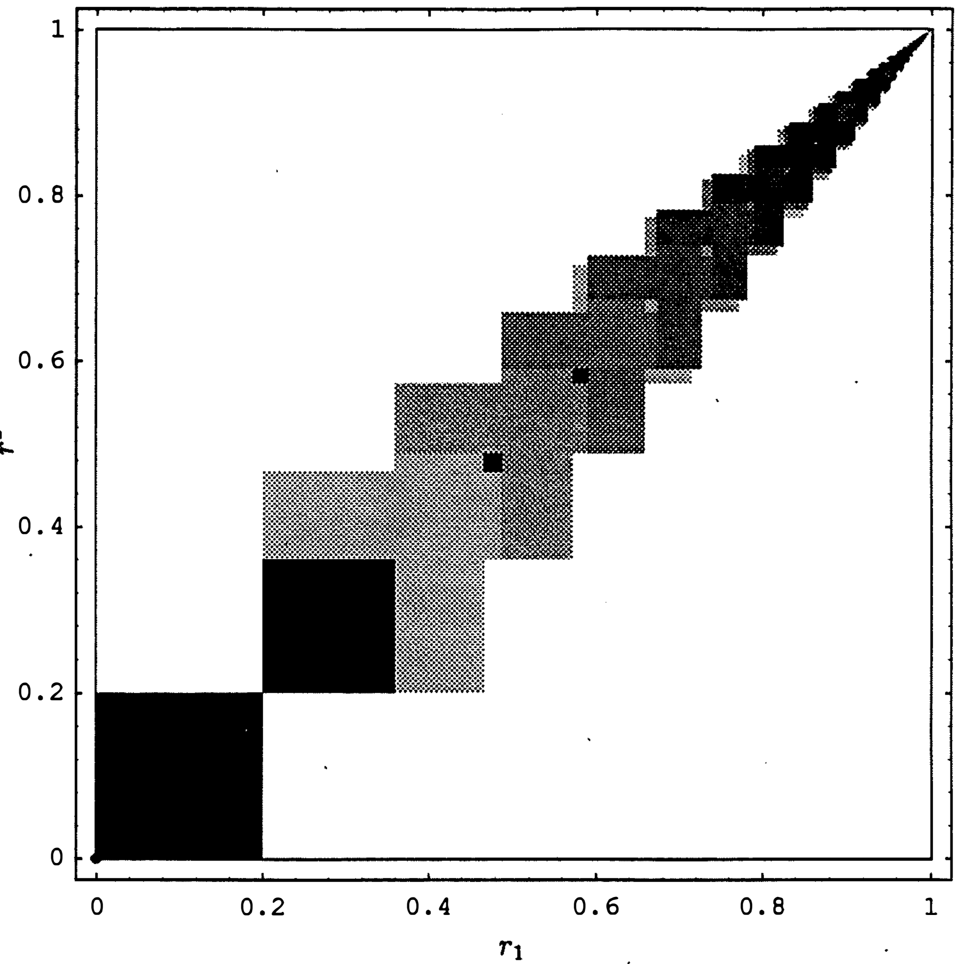


Figure 2. Graph of equilibrium correspondence for $b = .5$, $\delta = .75$. Unique alternating move equilibrium in white region. Unique simultaneous move equilibrium in black square areas. Shaded areas have $J$ stages of simultaneous move followed by alternating move. Darker shaded areas indicate larger $J$.

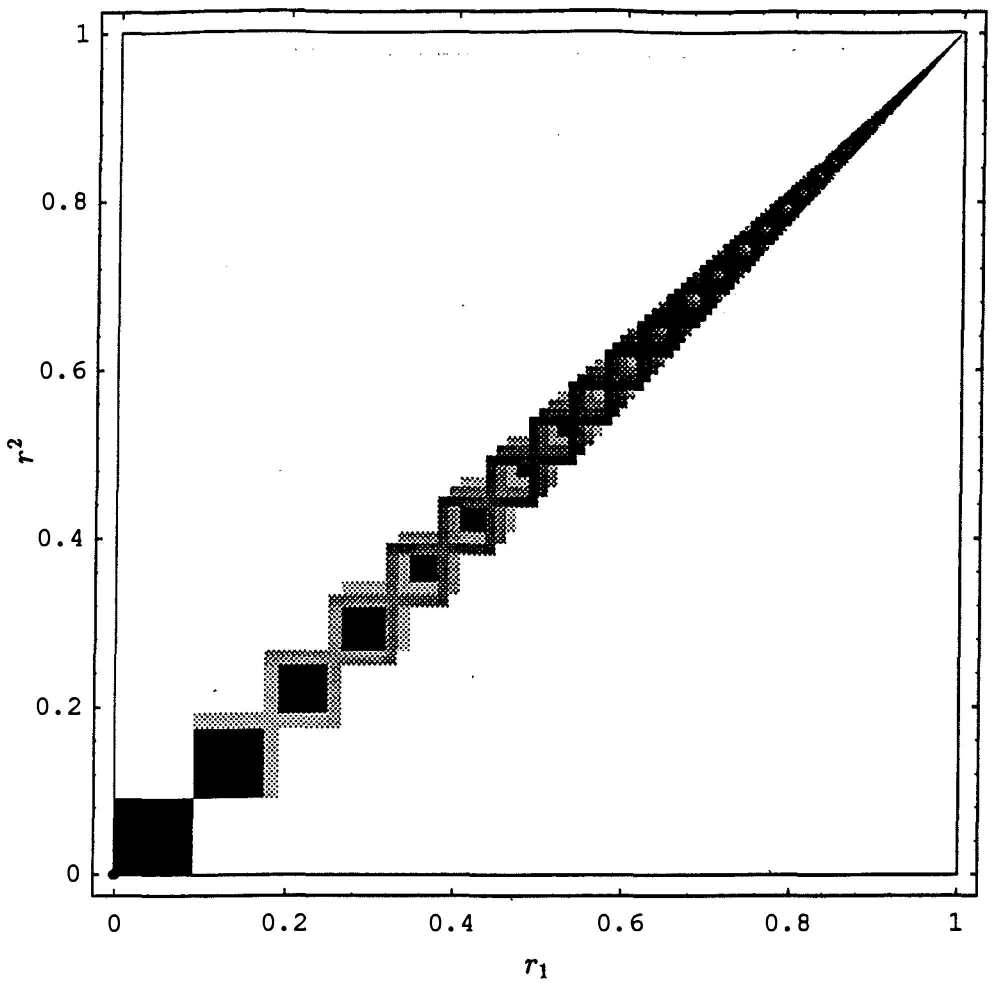


Figure 3. Graph of equilibrium correspondence for $b = .5$, $\delta = .9$. See Figure 2 for description.